

# A GENERALIZATION OF KOZLOV'S THEOREM ON INTEGRABLE MECHANICAL SYSTEMS ON SURFACES

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ABSTRACT. Let  $\mathfrak{S}$  be a compact, connected surface and  $H \in C^2(T^*\mathfrak{S})$  a Tonelli Hamiltonian. This note extends V. V. Kozlov's result on the Euler characteristic of  $\mathfrak{S}$  when  $H$  is real-analytically integrable, using a definition of topologically-tame integrability called semisimplicity. Theorem: If  $H$  is 2-semisimple, then  $\mathfrak{S}$  has non-negative Euler characteristic; if  $H$  is 1-semisimple and reversible, then  $\mathfrak{S}$  has positive Euler characteristic.

## 1. INTRODUCTION

**1.1. Kozlov's Theorem.** Say that a *natural* mechanical system is a Hamiltonian that is a sum of kinetic and potential energies. Let  $\mathfrak{S}$  be a compact surface and  $H : T^*\mathfrak{S} \rightarrow \mathbf{R}$  be an analytic natural mechanical system. Kozlov proved if  $H$  enjoys a second, independent analytic integral  $F$ , then the Euler characteristic of  $\mathfrak{S}$  is non-negative and so it is homeomorphic to  $\mathbf{S}^2$ ,  $\mathbf{T}^2$  or a non-orientable quotient thereof [11]. The hypotheses of Kozlov's theorem can be relaxed as follows: (i)  $H$  need only be assumed to be fibre-wise strictly convex and super-linear (i.e. 'Tonelli'); (ii) analyticity can be reduced to the combined hypotheses that  $H$  and  $F$  are  $C^2$ , and that there is an energy level  $H^{-1}(c)$  where  $c > \min \{H(x, 0) \mid x \in \mathfrak{S}\}$ , such that the critical set of  $F$  intersects a fibre of the foot-point projection in only finitely many points [12].

The present note has two aims. First, it presents a proof of Kozlov's theorem based on the theory of semisimplicity developed in [3]; see definition 1 below. Second, it extends Kozlov's theorem to non-commutatively integrable Tonelli Hamiltonians. The latter is a non-trivial extension: in [4] there are Tonelli Hamiltonians that are constructed which are non-commutatively integrable and semisimple on the unit disk bundle, but which are *not* tangent to a semisimple singular Lagrangian fibration. In essence, the naïve trick of discarding extra integrals to achieve complete integrability necessarily expands the critical set, and in the above-quoted example, the critical set expands from a real-analytic set to a wild set analogous to the Fox wild arc.

**1.2. Non-commutative integrability.** Let  $\Sigma$  be a smooth  $n$ -dimensional manifold. The canonical Poisson structure on the cotangent bundle  $T^*\Sigma$  permits one to define a Poisson algebra structure on  $C^\infty(T^*\Sigma)$  and consequently each smooth function  $H : T^*\Sigma \rightarrow \mathbf{R}$  induces a Hamiltonian vector field  $X_H$  defined by

$$(1) \quad X_H = \{H, \cdot\} = \sum_{i=1}^n -\frac{\partial H}{\partial x^i} \frac{\partial}{\partial y^i} + \frac{\partial H}{\partial y^i} \frac{\partial}{\partial x^i},$$

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where  $(x^i, y^i)$  are canonical coordinates. A first integral of the Hamiltonian vector field  $X_H$  is a smooth function  $F$  which Poisson commutes with  $H$ :  $\{H, F\} = 0$ .

For a subspace  $\mathfrak{A} \subset C^\infty(T^*\Sigma)$  and  $p \in T^*\Sigma$ , let  $d_p\mathfrak{A} = \text{span}\{df(p) \mid f \in \mathfrak{A}\}$ . Following [2], the differential dimension of  $\mathfrak{A}$  is defined to be  $\sup_p \dim d_p\mathfrak{A}$ . Let  $\mathfrak{A} \subset C^\infty(T^*\Sigma)$  be a subspace of first integrals of  $H$  that contains  $H$  and let  $Z(\mathfrak{A})$  be the subspace of  $\mathfrak{A}$  which Poisson commutes with all of  $\mathfrak{A}$ . Let  $k$  (resp.  $l$ ) be the differential dimension of  $\mathfrak{A}$  (resp.  $Z(\mathfrak{A})$ ). Say that a point  $p \in T^*\Sigma$  is *regular* for  $\mathfrak{A}$  if  $\dim d_q\mathfrak{A} = k$  and  $\dim d_q Z(\mathfrak{A}) = l$  for all  $q$  in the  $\mathfrak{A}$ -level set passing through  $p$ . We say that  $H$  (or  $\mathfrak{A}$ ) is *non-commutatively integrable* if  $k + l = 2n$  and the set of regular points is dense.

Nehorošev [15], who generalized the Liouville-Arnol'd theorem [1], proved that if  $H$  is non-commutatively integrable and  $p$  is a regular point, then there is a neighbourhood  $U$  with coordinate chart  $(\theta, I, x, y) : U \rightarrow \mathbf{T}^k \times \mathbf{R}^k \times \mathbf{R}^{2(n-k)}$ , where the Poisson bracket is canonical and  $H = H(I)$ . Dazord and Delzant [5] globalized this result and showed that the regular point set  $X \subset T^*\Sigma$  is fibred by isotropic  $k$ -dimensional tori  $F = \mathbf{T}^k$  and the quotient of  $X$  by these fibres,  $P$ , is a Poisson manifold with a foliation  $\zeta$  by symplectic leaves. When this foliation is a fibration, one has the following diagram

$$(2) \quad \begin{array}{ccccc} F & \hookrightarrow & X & \xhookrightarrow{\iota_X} & T^*\Sigma \\ & & \downarrow g & \searrow G & \\ S & \hookrightarrow & P & \xrightarrow{\pi} & Q \end{array}$$

where  $g$  is the fibration map,  $S$  is a symplectic leaf of  $P$ ,  $Q$  is an integral affine manifold of dimension  $k$  and  $\pi^*C^\infty(Q)$  is the centre of  $C^\infty(P)$  which induces the Hamiltonian vector fields that are tangent to the isotropic fibres of  $g$ .

**1.3. Geometric semisimplicity.** Let us abstract the notion of complete and non-commutative integrability. A smooth flow  $\varphi : M \times \mathbf{R} \rightarrow M$  is *integrable* if there is an open, dense subset  $R \subset M$  that is covered by angle-action charts which conjugate  $\varphi$  to a translation-type flow on the tori of  $\mathbf{T}^k \times \mathbf{R}^l$ . There is an open dense subset  $L \subset R$  fibred by  $\varphi$ -invariant tori; let  $f : L \rightarrow B$  be the induced smooth quotient map and let  $\Gamma = M - L$  be the *singular set*. If  $\Gamma$  is a tamely-embedded polyhedron, then  $\varphi$  is said to be *k-semisimple* with respect to  $(f, L, B)$ , or just semisimple [3]. Of most interest is when  $\varphi$  is a Hamiltonian flow on a cotangent bundle or possibly a regular iso-energy surface.

**Definition 1** (c.f. [17, 3]). *A Hamiltonian flow is geometrically k-semisimple if it is k-semisimple with respect to  $(f, L, B)$  and  $f$  is an isotropic fibration.*

In this case, because the fibres of  $f$  are isotropic,  $\varphi$  is non-commutatively integrable, so geometric semisimplicity is a topologically-tame type of non-commutative integrability. Taimanov [17] introduced a related notion of geometric simplicity, see sections 2.2-2.3 of [3] for further discussion. If  $\varphi$  is real-analytically non-commutatively integrable, then the triangulability of real-analytic sets implies that  $\varphi$  is geometrically semisimple; and  $B$  may be taken to be a disjoint union of open balls. On the other hand, geometric semisimplicity is a weaker property than real-analytic non-commutative integrability [3]. A basic question is:

**Question A.** *what are the obstructions to the existence of a geometrically semisimple (resp. semisimple, completely integrable) flow?*

**1.4. Main Results.** Here are the two main theorems of this note. In both cases,  $\mathfrak{S}$  is a compact, connected surface and  $H : T^*\mathfrak{S} \rightarrow \mathbf{R}$  is a  $C^2$  Tonelli Hamiltonian.

**Theorem 1** (*c.f.* Kozlov [11]). *If  $H$  is geometrically 1- or 2-semisimple, then  $\mathfrak{S}$  has non-negative Euler characteristic.*

Let us say that  $H$  is *reversible* if  $H \circ R = H$  where  $R : T^*\mathfrak{S} \rightarrow T^*\mathfrak{S}$  is the anti-symplectic involution  $(x, p) \mapsto (x, -p)$ .

**Theorem 2.** *If  $H$  is geometrically 1-semisimple and reversible, then  $\mathfrak{S}$  is homeomorphic to  $\mathbf{S}^2$  or  $\mathbf{RP}^2$ .*

This Theorem should be true without the reversibility hypothesis. The following remarks elaborate on this thought.

**Remark I.** The proof of Theorem 2 uses a theorem of Glasmachers & Knieper which characterises properties of geodesic flows on  $\mathbf{T}^2$  with vanishing topological entropy; they state that their results extend naturally to reversible Finsler metrics [7, 8]. It is uncertain if their results extend to non-reversible Finsler metrics; Knieper informs the author that he believes that they may not [10].

**Remark II.** Here is a proof of Theorem 1 alternative to the proof below. Below, in Lemma 3.1 it is proven that semisimplicity implies the topological entropy of the Hamiltonian flow of  $H$  must vanish. The Dinaburg-Manning theorem on the positivity of topological entropy for a geodesic flow on a compact surface of negative Euler characteristic (this theorem is true for Tonelli Hamiltonians with essentially no change) [6, 14], along with the result of Katok that a volume-preserving flow on a compact 3-manifold with positive topological entropy has a horseshoe [9], implies the Euler characteristic of  $\mathfrak{S}$  is non-negative. This paper's proof of Theorem 1 avoids an appeal to either of these results. It is this author's belief that Theorem 2 has a proof that does not use Katok's result, but this is elusive. Instead, this paper uses an argument similar to that of Paternain [16].

**Remark III.** Bangert has asked a series of questions in [13] concerning integrable Tonelli Hamiltonians which are integrable in a weaker sense—the additional integral need only be independent of  $H$  on an open dense set. These questions are very interesting but beyond the scope and techniques of this paper.

## 2. PRELIMINARIES

Let us recall a few items concerning a geometric semisimplicity. Let  $\Sigma$  be a compact smooth manifold and  $H : T^*\Sigma \rightarrow \mathbf{R}$  be a  $C^2$  Tonelli Hamiltonian that is geometrically semisimple with respect to  $(f, L, B)$ . The complement  $\Gamma = T^*\Sigma - L$  is a tamely embedded polyhedron, so the number of components of  $L \cap H^{-1}((-\infty, c])$  is finite for any  $c$ . [3, Lemma 15] implies that there is a component  $L_i \subset L$  such that  $\pi \iota_{L_i}$  has a finite-index image in  $\pi_1(\Sigma)$ :

$$(3) \quad \begin{array}{ccccc} & & \xrightarrow{\pi_* \iota_{L_i, *}} & & \\ & \nearrow & & \searrow & \\ \pi_1(L_i) & \xrightarrow{\iota_{L_i, *}} & \pi_1(T^*\Sigma) & \xrightarrow{\pi_*} & \pi_1(\Sigma). \end{array}$$

Suppose that  $B_0 \subset B$  is a nowhere dense subset such that  $f^{-1}(B_0) \cup \Gamma = \Gamma_1$  is tamely embedded polyhedron whose complement  $L_1 = f^{-1}(B_1)$ ,  $B_1 = B - B_0$ , is dense. One calls  $(f_1 = f|_{L_1}, L_1, B_1)$  a refinement of  $(f, L, B)$ . In [3, Lemma 18] it is proven that

**Proposition 2.1.** *If  $\dim B \leq 2$ , then  $(f, L, B)$  has a refinement  $(f_1, L_1, B_1)$  such that each component of  $B_1$  is homotopy equivalent to either a point or  $\mathbf{S}^1$ .*

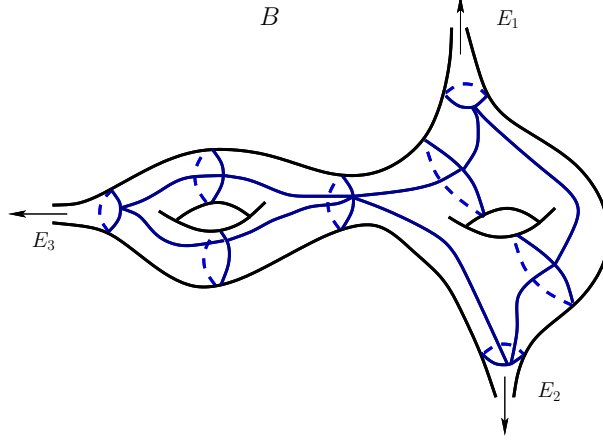


FIGURE 1. Schematic proof of 2.1: We cut the base  $B$  along the blue curves; the ends  $E_j$  are cylinders and the “compact” part of  $B_1$  is a union of disks.

### 3. PROOFS

Proposition 2.1 allows us to prove Theorem 1. The manifold  $\Sigma$  in the previous section is the surface  $\mathfrak{S}$ .

*Theorem 1.* Suppose that  $H$  is geometrically 2-semisimple. We will deal with the case of 1-semisimplicity below. By Proposition 2.1 we can suppose that each component of the base of the fibration  $f$  is homotopy equivalent to a point or a circle. Thus, each component of  $L_i \subset L$  is homotopy equivalent to  $\mathbf{T}^2$  or a  $\mathbf{T}^2$ -bundle over  $\mathbf{T}^1$ . In both cases,  $\pi_1(L_i)$  is solvable, and so  $\pi_1(\mathfrak{S})$  contains a solvable subgroup of finite index. Since  $\mathfrak{S}$  is a surface, the theorem is proved.  $\square$

*Theorem 2.* To prove Theorem 2, we must adapt the diagram in (2) to our needs. In this case, the fibre  $F = \mathbf{T}^1$ , the base of the fibration  $P(= B)$  is a 3-dimensional Poisson manifold with a foliation  $\zeta$  by symplectic surfaces  $S$ . The foliation  $\zeta$  is a fibration, in fact, because the Casimirs of  $P$  are functionally dependent on the reduction of the Hamiltonian  $H|X$ . In this case, the quotient of  $P$  by  $\zeta$ ,  $Q$ , is a finite union of 1-manifolds:  $Q = \cup_i Q_i$  where  $Q_i \simeq \mathbf{R}$  or  $\mathbf{T}^1$ . Since  $H|X = h \circ G$  for some  $h \in C^2(Q)$ , the Tonelli property of  $H$  implies that no component of  $Q$  is a circle.

It follows that there is a component  $X_i = G^{-1}(Q_i)$  such that

$$(4) \quad \begin{array}{ccccc} & & \pi_* \iota_{X_i, *} & & \\ & \nearrow & & \searrow & \\ \pi_1(X_i) & \xrightarrow{\iota_{X_i, *}} & \pi_1(T^*\mathfrak{S}) & \xrightarrow{\pi_*} & \pi_1(\mathfrak{S}) \end{array}$$

has a finite index image. Since  $X_i$  is homotopy equivalent to an  $F = \mathbf{T}^1$ -principal bundle over the symplectic surface  $S_i$ , a leaf of  $\zeta|X_i$ , it remains to examine the possibilities.

**$S_i$  is compact.** In this case,  $G^{-1}(q)$  is compact for any  $q \in Q_i$ , and therefore it must be a connected component of an energy level. Since, above the critical value, the energy levels are connected,  $G^{-1}(q)$  is an energy level. If the Euler characteristic of  $\mathfrak{S}$  is negative, then  $\pi_1(\mathfrak{S})$  contains no non-trivial normal abelian subgroups. Therefore, the inclusion  $F \hookrightarrow T^*\mathfrak{S}$  is null-homotopic; this implies that all orbits of the Tonelli Hamiltonian in a super-critical energy surface are contractible—absurd. Therefore, the Euler characteristic of  $\mathfrak{S}$  must be non-negative.

Since every orbit of the Tonelli Hamiltonian is closed on a super-critical energy level, the Euler characteristic of  $\mathfrak{S}$  must be positive.

**$S_i$  is non-compact.** Let  $c$  be an energy level such that  $\pi(S_i) = G(H^{-1}(c)) = q$ . Let  $X_c = G^{-1}(q)$ ,  $g_c = g|_{X_c}$  and  $P_c = g(X_c)$ .  $X_c \subset H^{-1}(c)$  has a complement  $\Gamma_c = \Gamma \cap H^{-1}(c)$  and is fibred by  $F = \mathbf{T}^1$ . The Hamiltonian flow of  $H$  restricted to  $H^{-1}(c)$  is therefore 1-semisimple with respect to  $(g_c, X_c, P_c)$ . Now,  $P_c$  is a symplectic leaf of the foliation  $\zeta$  and therefore is a disjoint union of connected symplectic surfaces  $S_j$ . By Proposition 2.1, there is a refinement  $(g'_c, X'_c, P'_c)$  such that each component of  $P'_c$  is homotopy equivalent to a point or  $\mathbf{T}^1$ . Moreover, by [3, Lemma 15], the inclusion of one of the components of  $X'_c$  in  $T^*\mathfrak{S}$  is almost surjective on  $\pi_1$ . But the components of  $X'_c$  are homotopy equivalent to  $\mathbf{T}^1$  or  $\mathbf{T}^2$  (principal  $\mathbf{T}^1$ -bundles over  $*$  and  $\mathbf{T}^1$  respectively).

Therefore,  $\pi_1(\mathfrak{S})$  contains a finite-index abelian subgroup. Hence the Euler characteristic of  $\mathfrak{S}$  is non-negative (this completes the proof of Theorem 1). It therefore remains to prove that the Euler characteristic of  $\mathfrak{S}$  is positive. To do so, we will prove

**Lemma 3.1.** *If  $H$  is 1- or 2-semisimple, then the Hamiltonian flow of  $H$  has zero topological entropy.*

and

**Lemma 3.2.** *Assume that  $H$  is 1-semisimple and reversible. Let  $c > \max \{H(p) \mid p \in 0(\mathfrak{S})\}$ ,  $q \in \mathfrak{S}$ ,  $S_{q,c} = H^{-1}(c) \cap T_q^*\mathfrak{S}$  and let  $r_{q,c} : S_{q,c} \rightarrow \mathbf{T}^1$  be the map that sends  $p \in S_{q,c}$  to the asymptotic rotation angle of the integral curve through  $p$ . Then, there is a  $q$  and  $c$  such that  $r_{q,c}$  is constant.*

Lemma 3.2 contradicts [7, Theorem 2], which asserts that the map  $r_{q,c}$  has degree 1 for all  $q \in \mathfrak{S}$  and all  $c$  above the critical value. This proves Theorem 2 modulo these two lemmata.

**Lemma 3.1.** Suppose that  $h_{top}(\phi_E) > 0$  for the Hamiltonian flow of  $H$  restricted to some energy level  $E$ . Katok's theorem on area-preserving diffeomorphisms of a surface with positive topological entropy implies that  $\phi_E$  has a horseshoe and so therefore does  $\phi_{E'}$  for all nearby energy levels [9]. These horseshoes must be contained in the singular  $\Gamma$ , but  $\Gamma \cap E''$  is a tamely embedded polyhedron of dimension at most 2 for a generic energy level  $E''$ . This is a contradiction.  $\square$

**Lemma 3.2.** First, let us observe that if  $\Gamma \cap S_{q,c} = S_{q,c}$  for all  $q$ , then the energy level  $H^{-1}(c) \subset \Gamma$ . If there is an interval  $I \ni c$  such that  $H^{-1}(c') \subset \Gamma$  for all  $c' \in I$ , then  $\Gamma$  has a non-empty interior. Absurd. Thus, there is a  $c'$  arbitrarily close to  $c$  such that  $\Gamma \cap S_{q,c} \not\subset \Gamma$  for some  $q \in \mathfrak{S}$ . Wolog, let us suppose that  $c = c'$ .

By a similar argument, we can suppose that  $\Gamma \cap S_{q,c}$  has an empty interior in  $S_{q,c}$ .

The embedding  $\phi : \mathbf{T}^1 \hookrightarrow S_{q,c}$  is smooth, and so there is a PL embedding  $\phi' : \mathbf{T}^1 \hookrightarrow H^{-1}(c)$  homotopic to  $\phi$  and transverse to  $\Gamma$ . Let  $S'_{q,c}$  denote the image of  $\phi'$  and let  $r'_{q,c} : S'_{q,c} \rightarrow \mathbf{T}^1$  denote the map that sends the initial condition  $p \in S'_{q,c}$  to the rotation angle of the curve  $\gamma_p(t) = \pi \circ \varphi_t(p)$  where  $\varphi$  is the Hamiltonian flow of  $H$ .

Since  $\Gamma$  is transverse to  $\phi'$ ,  $\Gamma \cap S'_{q,c}$  is a finite number of points. The complement  $L'_{q,c} = L \cap S'_{q,c}$  consists of a finite union of open intervals. On each component of  $L'_{q,c}$ , the orbits of  $\varphi$  are periodic. Due to the existence of angle-action variables, the orbits are therefore homotopic up to a reparameterisation of time. This shows that  $r'_{q,c}$  is constant on the components of  $L'_{q,c}$ .

On the other hand,  $r'_{q,c}$  is continuous by the continuity of the rotation map  $r : H^{-1}(c) \rightarrow \mathbf{T}^1$  [7]. Therefore, it is constant.

But,  $r'_{q,c}$  is homotopic to  $r_{q,c}$ , so  $r_{q,c}$  is constant.  $\square$

As noted above, this proves that the 1-semisimple, reversible Hamiltonian  $H$  cannot have a zero Euler characteristic configuration space. Therefore  $\mathfrak{S}$  is either  $\mathbf{S}^2$  or  $\mathbf{RP}^2$ .  $\square$

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